数理解析研究所講究録 第2109巻 2019年 115-127

The interactions of dark line solitons in the Davey-Stewartson II system

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Abstract

We study the interactions of dark line solitons of the defocusing Davey-Stewartson II (DS II) system. The angle dependency of dark line soliton interactions is investigated in detail by using an analytical method based on exact solutions and numerical experiments. The general multi-dark line soliton solution with full parameters is given in the Wronskian form and the chord diagrams for the DS II dark line solitons are presented.

1 Introduction

The studies of soliton interactions of two-dimensional soliton systems have been attracted many researchers over the last few decades. Recently, the line soliton interactions of the Kadomtsev-Petviashvili II (KP II) equation

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \tag{1}$$

has been studied in detail by Kodama and his collaborators [1–10].

The Davey-Stewartson (DS) system

$$\mathrm{i}u_t + u_{xx} - \sigma_1 u_{yy} + 2\sigma_2|u|^2u + 4uQ = 0, \tag{2}$$

$$Q_{xx} + \sigma_1 Q_{yy} + \sigma_2(|u|^2)_{xx} = 0, \tag{3}$$

is known as a mathematical model which describes two-dimensional water waves in finite depth [11–16]. Here the parameters σ_1 and σ_2 take $+1$ or -1. The case of $\sigma_1 = -1$ corresponds to the Davey-Stewartson I (DS I) system, and the case of $\sigma_1 = +1$ corresponds to the Davey-Stewartson II (DS II) system [15, 16]. The parameter σ_2 determines focusing ($\sigma_2 = -1$) or defocusing ($\sigma_2 = 1$).

The DS system is sometimes expressed in the following form:

$$\mathrm{i}u_{\tilde{t}} + \sigma_1 u_{xx} - u_{yy} + 2\sigma_3|u|^2u + 4\sigma_1 u\phi_x = 0\,, \tag{4}$$

$$\sigma_1\phi_{xx} + \phi_{yy} + \sigma_3(|u|^2)_x = 0\,. \tag{5}$$

(4) and (5) can be transformed into (2) and (3) by setting $Q = \phi_x$, $\sigma_2 = \sigma_1\sigma_3$, $t = \sigma_1\tilde{t}$.

The DS system (2) and (3) has a plane wave solution

$$u = \rho_0 e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})}, \quad Q = 0, \quad \omega = k^2 - \sigma_1 l^2 - 2\sigma_2\rho_0^2. \tag{6}$$

Here we look for soliton solutions with non-zero background, i.e., soliton solutions on the above plane wave. By the transformation of dependent variables

$$u=\rho_0\frac{g(x,y,t)}{f(x,y,t)}e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})},\quad u^*=\rho_0\frac{g^*(x,y,t)}{f(x,y,t)}e^{-\mathrm{i}(kx+ly-\omega t+\xi^{(0)})},\quad Q=\phi_x=(\log f(x,y,t))_{xx},\quad (7)$$

the bilinear equations of the DS system (2) and (3) are obtained:

$$\left(\mathrm{i}D_t+D_x^2-\sigma_1 D_y^2+2\mathrm{i}kD_x-2\mathrm{i}\sigma_1 lD_y\right)g\cdot f=0, \quad (8)$$

$$\left(-\mathrm{i}D_t+D_x^2-\sigma_1 D_y^2-2\mathrm{i}kD_x+2\mathrm{i}\sigma_1 lD_y\right)g^*\cdot f=0, \quad (9)$$

$$\left(D_x^2+\sigma_1 D_y^2+\omega-k^2+\sigma_1 l^2\right)f\cdot f+2\sigma_2\rho_0^2|g|^2=0, \quad (10)$$

where f is a real function, g is a complex function and g^* is its complex conjugate. The parameters k, l, ω, ρ_0, $\xi^{(0)}$ are real. The Hirota's D operators D_x, D_y and D_t are defined by

$$D_x^{m_x}D_y^{m_y}D_t^{m_t}f\cdot g=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^{m_x}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^{m_y}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^{m_t}f(x,y,t)g(x',y',t')\Big|_{x'=x,y'=y,t'=t}, \quad (11)$$

where m_x,m_y,m_t are positive integers.

By using the Hirota direct method, we obtain the 1-soliton solution

$$\begin{aligned}
&f=1+e^{px+qy-\Omega t+\theta^{(0)}},\quad g=1+\alpha e^{px+qy-\Omega t+\theta^{(0)}},\quad g^*=1+\alpha^* e^{px+qy-\Omega t+\theta^{(0)}}, \qquad (12)\\
&\alpha=\frac{2kp-2\sigma_1 lq+\mathrm{i}(p^2-\sigma_1 q^2)-\Omega}{2kp-2\sigma_1 lq-\mathrm{i}(p^2-\sigma_1 q^2)-\Omega},\quad \alpha^*=\frac{2kp-2\sigma_1 lq-\mathrm{i}(p^2-\sigma_1 q^2)-\Omega}{2kp-2\sigma_1 lq+\mathrm{i}(p^2-\sigma_1 q^2)-\Omega},\\
&\Omega=2kp-2\sigma_1 lq+\epsilon_1\frac{|p^2-\sigma_1 q^2|}{p^2+\sigma_1 q^2}\sqrt{(p^2+\sigma_1 q^2)(4\sigma_2\rho_0^2-p^2-\sigma_1 q^2)},\quad \epsilon_1=\pm1,\\
&\omega=k^2-\sigma_1 l^2-2\sigma_2\rho_0^2,
\end{aligned}$$

where $p,q,\theta^{(0)}$ are real parameters, and the 2-soliton solution

$$\begin{aligned}
&f=1+e^{\theta_1}+e^{\theta_2}+A_{12}e^{\theta_1+\theta_2}, \qquad (13)\\
&g=1+\alpha_1 e^{\theta_1}+\alpha_2 e^{\theta_2}+\alpha_1\alpha_2 A_{12}e^{\theta_1+\theta_2},\quad g^*=1+\alpha_1^* e^{\theta_1}+\alpha_2^* e^{\theta_2}+\alpha_1^*\alpha_2^* A_{12}e^{\theta_1+\theta_2},\\
&\theta_j=p_jx+q_jy-\Omega_jt+\theta_j^0,\quad \omega=k^2-\sigma_1 l^2-2\sigma_2\rho_0^2,\\
&\alpha_j=\frac{2kp_j-2\sigma_1 lq_j+\mathrm{i}(p_j^2-\sigma_1 q_j^2)-\Omega_j}{2kp_j-2\sigma_1 lq_j-\mathrm{i}(p_j^2-\sigma_1 q_j^2)-\Omega_j},\quad \alpha_j^*=\frac{2kp_j-2\sigma_1 lq_j-\mathrm{i}(p_j^2-\sigma_1 q_j^2)-\Omega_j}{2kp_j-2\sigma_1 lq_j+\mathrm{i}(p_j^2-\sigma_1 q_j^2)-\Omega_j},\\
&\Omega_j=2kp_j-2\sigma_1 lq_j+\frac{|p_j^2-\sigma_1 q_j^2|}{p_j^2+\sigma_1 q_j^2}\gamma_j,\quad \gamma_j=\epsilon_j\sqrt{(p_j^2+\sigma_1 q_j^2)(4\sigma_2\rho_0^2-p_j^2-\sigma_1 q_j^2)},\quad \epsilon_j=\pm1,\quad (j=1,2),\\
&A_{12}=\frac{4\sigma_2\rho_0^2(p_1p_2+\sigma_1 q_1q_2)-\gamma_1\gamma_2-(p_1^2+\sigma_1 q_1^2)(p_2^2+\sigma_1 q_2^2)}{4\sigma_2\rho_0^2(p_1p_2+\sigma_1 q_1q_2)-\gamma_1\gamma_2+(p_1^2+\sigma_1 q_1^2)(p_2^2+\sigma_1 q_2^2)}.
\end{aligned}$$

In this article, we study the dark-line soliton interactions of the defocusing DS II system, i.e., the case of $\sigma_1=+1,\sigma_2=+1$.

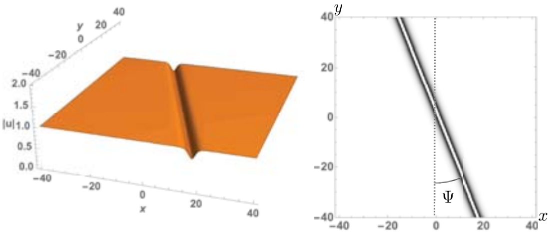


Figure 1: 1-dark line soliton of the DS II system.

2 Dark line soliton solutions of the defocusing Davey-Stewartson II system

Hereafter we consider the defocusing DS II system, i.e., the case of $\sigma_1 = +1, \sigma_2 = +1$. The dark-type line soliton solution of the defocusing DS II system is given by

$$
\begin{aligned}
u &= \rho_0 e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})}\frac{1+\alpha e^{px+qy-\Omega t+\theta^{(0)}}}{1+e^{px+qy-\Omega t+\theta^{(0)}}}, \qquad \omega = k^2 - l^2 - 2\rho_0^2, \\
\alpha &= \frac{2kp-2lq+\mathrm{i}(p^2-q^2)-\Omega}{2kp-2lq-\mathrm{i}(p^2-q^2)-\Omega}, \quad \alpha^* = \frac{2kp-2lq-\mathrm{i}(p^2-q^2)-\Omega}{2kp-2lq+\mathrm{i}(p^2-q^2)-\Omega}, \\
|u|^2 &= \rho_0^2 - \frac{p^2+q^2}{4}\mathrm{sech}^2\,\frac{px+qy-\Omega t+\theta^{(0)}}{2} \\
&= \rho_0^2 - A\,\mathrm{sech}^2\,\sqrt{A}[(\cos\Psi)x + (\sin\Psi)y - Ct + x^{(0)}], \\
Q &= \phi_x = \frac{p^2}{4}\mathrm{sech}^2\,\frac{px+qy-\Omega t+\theta^{(0)}}{2}, \\
\Omega &= 2kp - 2lq + \epsilon_1\frac{|p^2-q^2|}{p^2+q^2}\sqrt{(p^2+q^2)(4\rho_0^2-p^2-q^2)}, \\
A &= \frac{p^2+q^2}{4}, \quad \cos\Psi = \frac{p}{\sqrt{p^2+q^2}}, \quad \sin\Psi = \frac{q}{\sqrt{p^2+q^2}}, \quad x^{(0)} = \frac{\theta^0}{\sqrt{p^2+q^2}}, \\
C &= \frac{\Omega}{\sqrt{p^2+q^2}} = \frac{2kp-2lq}{\sqrt{p^2+q^2}} + \epsilon_1\frac{|p^2-q^2|}{p^2+q^2}\sqrt{4\rho_0^2-p^2-q^2}, \quad \epsilon_1 = \pm 1.
\end{aligned}
\tag{14}
$$

In the DS II line solitons, Ω and C can take 2 different values for the same set of parameters p, q, k, l, ρ_0 because of the parameter ϵ_1. If we set $k = l = 0$, these solitons for $\epsilon_1 = +1$ and $\epsilon_2 = -1$ propagate in the opposite direction. Note that line solitons in the KP II equation do not have this property and KP II line solitons always propagate in the same direction. In the 1-dark line soliton solution, $\tan\Psi = \frac{q}{p}$ corresponds to the slope of a line soliton and Ψ corresponds to an angle which is measured clockwise from the negative part of y-axis (see Fig.1). The depth of the dark line soliton is given by

$$
\sqrt{\rho_0^2 - \frac{p^2+q^2}{4}}. \tag{15}
$$

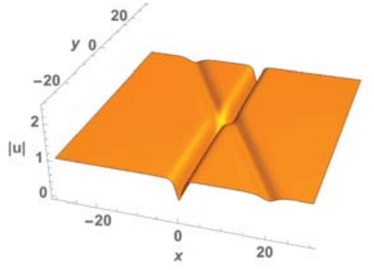


Figure 2: 2-dark line soliton of the DS II system.

The 2-dark line soliton solution of the defocusing DS II system is given by

$$u=\rho_0\frac{g(x,y,t)}{f(x,y,t)}e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})},\quad Q=(\log f(x,y,t))_{xx} \tag{16}$$

$$f=1+e^{\theta_1}+e^{\theta_2}+A_{12}e^{\theta_1+\theta_2},$$

$$g=1+\alpha_1e^{\theta_1}+\alpha_2e^{\theta_2}+\alpha_1\alpha_2A_{12}e^{\theta_1+\theta_2},\quad g^*=1+\alpha_1^*e^{\theta_1}+\alpha_2^*e^{\theta_2}+\alpha_1^*\alpha_2^*A_{12}e^{\theta_1+\theta_2},$$

$$\theta_j=p_jx+q_jy-\Omega_jt+\theta_j^0,\quad \omega=k^2-l^2-2\rho_0^2,$$

$$\alpha_j=\frac{2kp_j-2lq_j+\mathrm{i}(p_j^2-q_j^2)-\Omega_j}{2kp_j-2lq_j-\mathrm{i}(p_j^2-q_j^2)-\Omega_j},\quad \alpha_j^*=\frac{2kp_j-2lq_j-\mathrm{i}(p_j^2-q_j^2)-\Omega_j}{2kp_j-2lq_j+\mathrm{i}(p_j^2-q_j^2)-\Omega_j},$$

$$\Omega_j=2kp_j-2lq_j+\frac{|p_j^2-q_j^2|}{p_j^2+q_j^2}\gamma_j,\quad \gamma_j=\epsilon_j\sqrt{(p_j^2+q_j^2)(4\rho_0^2-p_j^2-q_j^2)},\quad \epsilon_j=\pm1,\quad (j=1,2)$$

$$A_{12}=-\frac{4\rho_0^2(p_1p_2+q_1q_2)-\gamma_1\gamma_2-(p_1^2+q_1^2)(p_2^2+q_2^2)}{4\rho_0^2(p_1p_2+q_1q_2)-\gamma_1\gamma_2+(p_1^2+q_1^2)(p_2^2+q_2^2)}.$$

The graph of the 2-dark line soliton is given in Fig.2.

In the above 1-dark line soliton solution, by setting

$$p=2\rho_0\cos\Psi\sin\Phi,\quad q=2\rho_0\sin\Psi\sin\Phi, \tag{17}$$

u, α, α^*, Ω and C are written as

$$u=\rho_0e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})}\frac{1+\alpha e^{(2\rho_0\cos\Psi\sin\Phi)x+(2\rho_0\sin\Psi\sin\Phi)y-\Omega t+\theta^{(0)}}}{1+e^{(2\rho_0\cos\Psi\sin\Phi)x+(2\rho_0\sin\Psi\sin\Phi)y-\Omega t+\theta^{(0)}}},\qquad \omega=k^2-l^2-2\rho_0^2, \tag{18}$$

$$\alpha=e^{-2\epsilon_1\mathrm{i}\Phi},\quad \alpha^*=e^{2\epsilon_1\mathrm{i}\Phi},$$

$$\Omega=4k\rho_0\cos\Psi\sin\Phi-4l\rho_0\sin\Psi\sin\Phi+2\epsilon_1\rho_0^2|\cos2\Psi\sin2\Phi|,\quad \epsilon_1=\pm1. \tag{19}$$

In this parametrization, the depth of the dark line soliton is given by

$$\sqrt{\rho_0^2-\frac{p^2+q^2}{4}}=\rho_0|\cos\Phi|, \tag{20}$$

so dark line solitons disappear if $\Phi=\frac{\pi}{2},\frac{3\pi}{2}$ (i.e. $p^2+q^2=4\rho_0^2$). In the case of $k=l=0$, dark line solitons corresponding to $\Psi=\frac{\pi}{4}(=45°)$, $\frac{3\pi}{4}(=135°)$, $\frac{5\pi}{4}=(225°)$, $\frac{7\pi}{4}(=315°)$ do not move (i.e, $C=0$) and change the direction of propagation at these critical angles. In the case of $\epsilon_1 = -1$, by setting $\Psi \to \Psi + \pi$ we have

$$
\begin{aligned}
& p \to 2\rho_0 \cos(\Psi+\pi) \sin\Phi = -p, \quad q \to 2\rho_0 \sin(\Psi+\pi)\sin\Phi = -q, \\
& \Omega_- \to 4k\rho_0 \cos(\Psi+\pi)\sin\Phi - 4l\rho_0 \sin(\Psi+\pi)\sin\Phi - 2\rho_0^2|\cos 2(\Psi+\pi)\sin 2\Phi| \\
& \qquad = -4k\rho_0 \cos\Psi \sin\Phi + 4l\rho_0 \sin\Psi\sin\Phi - 2\rho_0^2|\cos 2\Psi \sin 2\Phi| \\
& \qquad = -\Omega_+ \\
& u \to \rho_0 e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})} \frac{1+e^{2\mathrm{i}\Phi}e^{(2\rho_0\cos(\Psi+\pi)\sin\Phi)x+(2\rho_0\sin(\Psi+\pi)\sin\Phi)y-\Omega_- t+\theta^{(0)}}}{1+e^{(2\rho_0\cos(\Psi+\pi)\sin\Phi)x+(2\rho_0\sin(\Psi+\pi)\sin\Phi)y-\Omega_- t+\theta^{(0)}}} \\
& \qquad = \rho_0 e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})} \frac{1+e^{2\mathrm{i}\Phi}e^{-(2\rho_0\cos\Psi\sin\Phi)x-(2\rho_0\sin\Psi\sin\Phi)y-\Omega_- t+\theta^{(0)}}}{1+e^{-(2\rho_0\cos\Psi\sin\Phi)x-(2\rho_0\sin\Psi\sin\Phi)y-\Omega_- t+\theta^{(0)}}} \\
& \qquad = \rho_0 e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})} \frac{1+e^{-2\mathrm{i}\Phi}e^{(2\rho_0\cos\Psi\sin\Phi)x+(2\rho_0\sin\Psi\sin\Phi)y+\Omega_+ t-\theta^{(0)}}}{1+e^{(2\rho_0\cos\Psi\sin\Phi)x+(2\rho_0\sin\Psi\sin\Phi)y+\Omega_+ t-\theta^{(0)}}}
\end{aligned}
$$

where $\Omega_+ = \Omega$ for $\epsilon_1 = 1$ and $\Omega_- = \Omega$ for $\epsilon_1 = -1$. Thus the case of $\epsilon_1 = -1$ corresponds to 180°-rotation ($\Psi \to \Psi + \pi$) in the case of $\epsilon_1 = +1$. Thus we can rewrite the 1-dark line soliton solution in the following form without ϵ_1:

$$
\begin{aligned}
& u = \rho_0 e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})} \frac{1+\alpha e^{(2\rho_0\cos\Psi\sin\Phi)x+(2\rho_0\sin\Psi\sin\Phi)y-\Omega t+\theta^{(0)}}}{1+e^{(2\rho_0\cos\Psi\sin\Phi)x+(2\rho_0\sin\Psi\sin\Phi)y-\Omega t+\theta^{(0)}}}, \qquad \omega = k^2 - l^2 - 2\rho_0^2, \\
& \alpha = e^{-2\mathrm{i}\Phi}, \quad \alpha^* = e^{2\mathrm{i}\Phi}, \quad \Omega = 4k\rho_0\cos\Psi\sin\Phi - 4l\rho_0\sin\Psi\sin\Phi + 2\rho_0^2\cos 2\Psi\sin 2\Phi.
\end{aligned} \tag{21}
$$

In the 2-dark line soliton solution, by setting

$$
p_j = 2\rho_0\cos\Psi_j\sin\Phi_j, \quad q_j = 2\rho_0\sin\Psi_j\sin\Phi_j \quad (j=1,2) \tag{22}
$$

α_j, α_j^*, Ω_j $(j=1,2)$ and A_{12} are written as

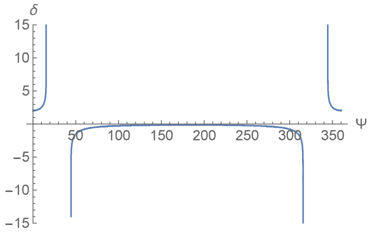
$$
\begin{aligned}
& \alpha_j = e^{-2\epsilon_j \mathrm{i}\Phi_j}, \quad \alpha_j^* = e^{2\epsilon_j\mathrm{i}\Phi_j}, \\
& \Omega_j = 4k\rho_0\cos\Psi_j\sin\Phi_j - 4l\rho_0\sin\Psi_j\sin\Phi_j + 2\epsilon_j\rho_0^2|\cos 2\Psi_j\sin 2\Phi_j|, \quad \epsilon_j = \pm 1 \quad (j=1,2) \\
& A_{12} = \frac{4\sin\Phi_1\sin\Phi_2(\cos(\Psi_1-\Psi_2) - \sin\Phi_1\sin\Phi_2) - \epsilon_1\epsilon_2|\sin 2\Phi_1||\sin 2\Phi_2|}{4\sin\Phi_1\sin\Phi_2(\cos(\Psi_1-\Psi_2) + \sin\Phi_1\sin\Phi_2) - \epsilon_1\epsilon_2|\sin 2\Phi_1||\sin 2\Phi_2|}.
\end{aligned} \tag{23}
$$

As in the 1-dark line soliton solution, the case of $\epsilon_j = -1$ corresponds to 180°-rotation ($\Psi_j \to \Psi_j + \pi$) in the case of $\epsilon_j = +1$. we can rewrite α_j, α_j^*, Ω_j $(j=1,2)$ and A_{12} in the following form without ϵ_1 and ϵ_2

$$
\begin{aligned}
& \alpha_j = e^{-2\mathrm{i}\Phi_j}, \quad \alpha_j^* = e^{2\mathrm{i}\Phi_j}, \quad \Omega_j = 4k\rho_0\cos\Psi_j\sin\Phi_j - 4l\rho_0\sin\Psi_j\sin\Phi_j + 2\rho_0^2\cos 2\Psi_j\sin 2\Phi_j, \quad (j=1,2) \\
& A_{12} = \frac{4\sin\Phi_1\sin\Phi_2(\cos(\Psi_1-\Psi_2) - \sin\Phi_1\sin\Phi_2) - \sin 2\Phi_1\sin 2\Phi_2}{4\sin\Phi_1\sin\Phi_2(\cos(\Psi_1-\Psi_2) + \sin\Phi_1\sin\Phi_2) - \sin 2\Phi_1\sin 2\Phi_2}.
\end{aligned} \tag{24}
$$

3 Angle dependency of dark line soliton interactions

Let us consider the angle dependency of the 2-dark line soliton interactions of the defocusing DS II system.



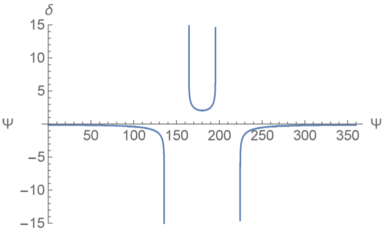


Figure 3: Angle dependency of 2-dark soliton interaction.

In the previous section, we obtained the following form of the 2-dark line soliton solution:

$$\begin{aligned}
&u = \rho_0 \frac{g(x,y,t)}{f(x,y,t)} e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})}, \quad Q = (\log f(x,y,t))_{xx} \qquad (25)\\
&f = 1 + e^{\theta_1} + e^{\theta_2} + A_{12} e^{\theta_1+\theta_2},\\
&g = 1 + \alpha_1 e^{\theta_1} + \alpha_2 e^{\theta_2} + \alpha_1\alpha_2 A_{12} e^{\theta_1+\theta_2}, \quad g^* = 1 + \alpha_1^* e^{\theta_1} + \alpha_2^* e^{\theta_2} + \alpha_1^*\alpha_2^* A_{12} e^{\theta_1+\theta_2},\\
&\theta_j = p_j x + q_j y - \Omega_j t + \theta_j^0, \quad \omega = k^2 - l^2 - 2\rho_0^2,\\
&p_j = 2\rho_0 \cos\Psi_j \sin\Phi_j, \quad q_j = 2\rho_0 \sin\Psi_j \sin\Phi_j, \quad \alpha_j = e^{-2\mathrm{i}\Phi_j}, \quad \alpha_j^* = e^{2\mathrm{i}\Phi_j},\\
&\Omega_j = 4k\rho_0 \cos\Psi_j \sin\Phi_j - 4l\rho_0 \sin\Psi_j \sin\Phi_j + 2\rho_0^2 \cos 2\Psi_j \sin 2\Phi_j, \quad (j=1,2)\\
&A_{12} = \frac{4\sin\Phi_1 \sin\Phi_2(\cos(\Psi_1-\Psi_2) - \sin\Phi_1\sin\Phi_2) - \sin 2\Phi_1 \sin 2\Phi_2}{4\sin\Phi_1 \sin\Phi_2(\cos(\Psi_1-\Psi_2) + \sin\Phi_1\sin\Phi_2) - \sin 2\Phi_1 \sin 2\Phi_2}.
\end{aligned}$$

Since A_{12} in the 2-dark line soliton solution determines the soliton interaction, we investigate the angle dependency of $\delta = -\log A_{12}$ which describes the phase shift by soliton interaction [17]. Note that A_{12} does not include the parameters k, l, ω. This means that a continuous wave background (a plane wave) does not affect dark-line soliton interactions.

Figure 3 shows the angle dependency of 2-dark soliton interactions. We consider the fixed vertical soliton and rotate another soliton in the counterclockwise from the negative part in y-axis. The parameters in the left graph are $p_1 = 1$, $q_1 = 0$, $p_2 = 2\cos\Psi_2$, $q_2 = 2\sin\Psi_2$ (i.e., $\rho_0 = 2$, $\Psi_1 = 0°$, $\sin\Phi_1 = \frac{1}{4}$, $\sin\Phi_2 = \frac{1}{2}$). In this case, the fixed vertical soliton propagates to right. The parameters in the right graph are $p_1 = -1$, $q_1 = 0$, $p_2 = 2\cos\Psi_2$, $q_2 = 2\sin\Psi_2$ (i.e., $\rho_0 = 2$, $\Psi_1 = 180°$, $\sin\Phi_1 = \frac{1}{4}$, $\sin\Phi_2 = \frac{1}{2}$). In this case, the fixed vertical soliton propagate to left. Note that the right graph is obtained from the left graph by the 180° shift of the angle. The right graph is obtained by the 180° shift of either Ψ_1 or Ψ_2.

In the case of the KP II equation, the region of $\delta > 0$ in the graph corresponds to $0 < A_{12} < 1$ in which the soliton interaction is called P-type, and the region of $\delta < 0$ in the graph corresponds to $A_{12} > 1$ in which the soliton interaction is called O-type. The region in which there is no curve corresponds to $A_{12} < 0$, thus the corresponding τ-function gives a singular solution. In these regions of Ψ_2, the T-type soliton solutions which have a hole in the intermediate region appear instead of the above 2-dark line soliton solution. The angle dependency of dark line soliton interactions of the DS II system is similar to that of the KP II line soliton interactions.

We have performed numerical experiments of 2-dark line soliton interactions. In our numerical experiments, we have employed the split-step Fourier method developed by White and Weideman [18]. To resolve inconsistency of boundary condition, we have used the windowing technique [6, 8, 19]. We have confirmed that the above theoretical prediction by the exact 2-soliton solution matches with numerical results. We will report the detail of numerical studies of the DS II dark line solitons in our forthcoming paper.

4 Chord diagrams of dark line soliton solutions of the defocusing Davey-Stewartson II system

In this section, we introduce a new parametrization for dark line soliton solutions of the defocusing DS II system.

1-dark line soliton solution

By setting

$$\Psi = \frac{\psi_2 + \psi_1}{2}, \quad \Phi = \frac{\psi_2 - \psi_1}{2}, \quad \psi_1 < \psi_2 \tag{26}$$

p, q, α, α^* and Ω are rewritten as

$$\begin{aligned}
&\Omega = 2k\rho_0(\sin\psi_2 - \sin\psi_1) + 2l\rho_0(\cos\psi_2 - \cos\psi_1) + \rho_0^2(\sin 2\psi_2 - \sin 2\psi_1),\\
&p = 2\rho_0\cos\Psi\sin\Phi = \rho_0(\sin\psi_2 - \sin\psi_1), \quad q = 2\rho_0\sin\Psi\sin\Phi = -\rho_0(\cos\psi_2 - \cos\psi_1),\\
&\alpha = e^{-\mathrm{i}(\psi_2-\psi_1)}, \quad \alpha^* = e^{\mathrm{i}(\psi_2-\psi_1)}.
\end{aligned} \tag{27}$$

Then the 1-dark line soliton solution is rewritten in the following form:

$$\begin{aligned}
&u = \rho_0\frac{g}{f}e^{\mathrm{i}(kx+ly-\omega t+\xi^{(0)})}, \quad Q = (\log f)_{xx}, \quad \omega = k^2 - l^2 - 2\rho_0^2, \qquad (28)\\
&f = e^{\theta_1} + ae^{\theta_2}, \quad g = e^{\theta_1 - \mathrm{i}\psi_1} + ae^{\theta_2 - \mathrm{i}\psi_2}, \quad a > 0,\\
&\theta_j = (\rho_0\sin\psi_j)x - (\rho_0\cos\psi_j)y - (2k\rho_0\sin\psi_j + 2l\rho_0\cos\psi_j + \rho_0^2\sin 2\psi_j)t, \quad (j = 1, 2)\\
&|u|^2 = \rho_0^2 - \rho_0^2\sin^2\frac{\psi_2 - \psi_1}{2}\mathrm{sech}^2\left[\frac{1}{2}(\theta_2 - \theta_1 + \log a)\right],\\
&Q = \rho_0^2\frac{(\sin\psi_2 - \sin\psi_1)^2}{4}\mathrm{sech}^2\left[\frac{1}{2}(\theta_2 - \theta_1 + \log a)\right], \quad \psi_1 < \psi_2.
\end{aligned}$$

Here we set $\epsilon_1 = +1$ since the case of $\epsilon_1 = -1$ is 180°-rotation of a dark line soliton in the case of $\epsilon_1 = 1$. Although a dark line soliton of the same shape can propagate in 2 different speeds, we can interpret that the one dark line soliton among these two dark line solitons with different speeds is 180°-rotation of the other dark line soliton.

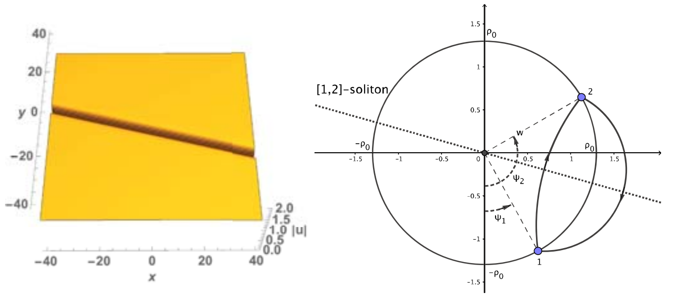


Figure 4: The graph of the dark line soliton (left) and its chord diagram (right).

Note that the parametrization using ψ_1 and ψ_2 provides us a simpler expression of dark line soliton solutions of the DS II system, and in this form we can find the important information for constructing chord diagrams of the DS II dark line soliton solutions. In Fig.4 we show the graph and the chord diagram of the 1-dark line soliton. To draw a chord diagram corresponding to a given 1-dark line soliton solution, we first draw a circle with the radius ρ_0 and points on the circle corresponding to angles ψ_1 and ψ_2, then connect

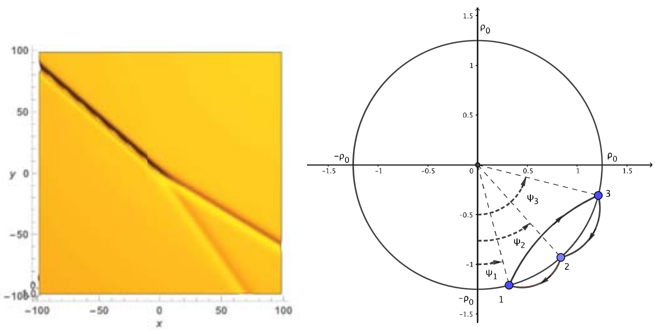


Figure 5: The graph of the resonant dark line soliton (left) and its chord diagram (right).

these points by arcs. Here parameters ψ_1 and ψ_2 are measured counterclockwise from the negative part in y-axis. The parameter $\Psi = \frac{\psi_2+\psi_1}{2}$ provides the location of the dark line soliton, and the dark line soliton corresponds to a perpendicular bisector of the line segment connecting 2 points on the circle. The other parameter $\Phi = \frac{\psi_2-\psi_1}{2}$ gives the depth of the dark soliton and this corresponds to the length of the line segment connecting 2 points on the circle. Since the 1-dark line soliton corresponds to the permutation of the index 1 in ψ_1 and the index 2 in ψ_2, i.e.,

$$\pi = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (1\,2),$$

we call it [1,2]-soliton.

Resonant Y-shape dark line soliton solution

The resonant Y-shape dark line soliton solution is given by

$$\begin{aligned}
&f = e^{\theta_1} + a_{12}e^{\theta_2} + a_{13}e^{\theta_3}, \quad g = e^{\theta_1 - \mathrm{i}\psi_1} + a_{12}e^{\theta_2 - \mathrm{i}\psi_2} + a_{13}e^{\theta_3 - \mathrm{i}\psi_3}, \quad a_{12}, a_{13} > 0, \\
&\theta_j = (\rho_0 \sin\psi_j)x - (\rho_0 \cos\psi_j)y - (2k\rho_0 \sin\psi_j + 2l\rho_0 \cos\psi_j + \rho_0^2 \sin 2\psi_j)t, \quad (j = 1, 2, 3) \\
&\psi_1 < \psi_2 < \psi_3.
\end{aligned} \tag{29}$$

As in the case of the KP II equation, this solution corresponds to the permutation $\pi = (3\,1\,2)$. The graph and the chord diagram of this resonant soliton solution are given in Fig.5. Setting wave numbers $p_i, q_i (i = 1, 2, 3)$ and angular frequencies $\Omega_i (i = 1, 2, 3)$ of dark line solitons as

$$\begin{aligned}
&p_1 = \rho_0(\sin\psi_2 - \sin\psi_1), \quad p_2 = \rho_0(\sin\psi_3 - \sin\psi_2), \quad p_3 = \rho_0(\sin\psi_3 - \sin\psi_1), \\
&q_1 = -\rho_0(\cos\psi_2 - \cos\psi_1), \quad q_2 = -\rho_0(\cos\psi_3 - \cos\psi_2), \quad q_3 = -\rho_0(\cos\psi_3 - \cos\psi_1), \\
&\Omega_1 = 2k\rho_0(\sin\psi_2 - \sin\psi_1) + 2l\rho_0(\cos\psi_2 - \cos\psi_1) + \rho_0^2(\sin 2\psi_2 - \sin 2\psi_1)), \\
&\Omega_2 = 2k\rho_0(\sin\psi_3 - \sin\psi_2) + 2l\rho_0(\cos\psi_3 - \cos\psi_2) + \rho_0^2(\sin 2\psi_3 - \sin 2\psi_2)), \\
&\Omega_3 = 2k\rho_0(\sin\psi_3 - \sin\psi_1) + 2l\rho_0(\cos\psi_3 - \cos\psi_1) + \rho_0^2(\sin 2\psi_3 - \sin 2\psi_1)),
\end{aligned} \tag{30}$$

we can find that resonant soliton condition

$$p_1 + p_2 = p_3, \quad q_1 + q_2 = q_3, \quad \Omega_1 + \Omega_2 = \Omega_3 \tag{31}$$

is satisfied. More complicated soliton resonance solutions which show soliton reconnection were discussed in [20].

2-dark line soliton solution

In the 2-dark line soliton solution, we introduce the new parametrization

$$\Psi_j = \frac{\psi_{2j}+\psi_{2j-1}}{2}, \quad \Phi_j = \frac{\psi_{2j}-\psi_{2j-1}}{2}, \quad (j=1,2), \quad \psi_1<\psi_2<\psi_3<\psi_4 \tag{32}$$

Then Ω_j, p_j, q_j, α_j, α_j^* $(j=1,2)$ and A_{12} are written by

$$\begin{aligned}
&\Omega_j = 2k\rho_0(\sin\psi_{2j}-\sin\psi_{2j-1}) + 2l\rho_0(\cos\psi_{2j}-\cos\psi_{2j-1}) + \rho_0^2(\sin 2\psi_{2j}-\sin 2\psi_{2j-1}),\\
&p_j = \rho_0(\sin\psi_{2j}-\sin\psi_{2j-1}), \quad q_j = -\rho_0(\cos\psi_{2j}-\cos\psi_{2j-1}),\\
&\alpha_j = e^{-\mathrm{i}(\psi_{2j}-\psi_{2j-1})}, \quad \alpha_j^* = e^{\mathrm{i}(\psi_{2j}-\psi_{2j-1})}, (j=1,2),\\
&A_{12} = \frac{B_- - \sin(\psi_2-\psi_1)\sin(\psi_4-\psi_3)}{B_+ - \sin(\psi_2-\psi_1)\sin(\psi_4-\psi_3)} = \frac{\sin\frac{\psi_3-\psi_1}{2}\sin\frac{\psi_4-\psi_2}{2}}{\sin\frac{\psi_3-\psi_2}{2}\sin\frac{\psi_4-\psi_1}{2}}\\
&B_+ = 4\sin\frac{\psi_2-\psi_1}{2}\sin\frac{\psi_4-\psi_3}{2}\left(\cos\frac{\psi_4+\psi_3-\psi_2-\psi_1}{2} + \sin\frac{\psi_2-\psi_1}{2}\sin\frac{\psi_4-\psi_3}{2}\right)\\
&B_- = 4\sin\frac{\psi_2-\psi_1}{2}\sin\frac{\psi_4-\psi_3}{2}\left(\cos\frac{\psi_4+\psi_3-\psi_2-\psi_1}{2} - \sin\frac{\psi_2-\psi_1}{2}\sin\frac{\psi_4-\psi_3}{2}\right)
\end{aligned} \tag{33}$$

The soliton interaction of this solution is called O-type and corresponds to the permutation $\pi = (2\,1\,4\,3)$. The chord diagram of the O-type 2-dark line soliton solution is given in Fig.6 (left figure).

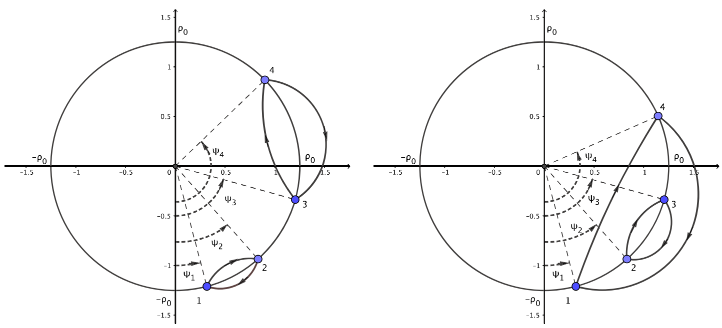


Figure 6: The chord diagrams of O-type (left) and P-type (right) 2-dark line soliton solutions.

In the 2-dark line soliton solution, we can set

$$\Psi_j = \frac{\psi_{5-j}+\psi_j}{2}, \quad \Phi_j = \frac{\psi_{5-j}-\psi_j}{2}, \quad (j=1,2), \quad \psi_1<\psi_2<\psi_3<\psi_4 \tag{34}$$

Ω_j, p_j, q_j, α_j, α_j^* $(j=1,2)$ and A_{12} are written by

$$\begin{aligned}
&\Omega_j = 2k\rho_0(\sin\psi_{5-j}-\sin\psi_j) + 2l\rho_0(\cos\psi_{5-j}-\cos\psi_j) + \rho_0^2(\sin 2\psi_{5-j}-\sin 2\psi_j),\\
&p_j = \rho_0(\sin\psi_{5-j}-\sin\psi_j), \quad q_j = -\rho_0(\cos\psi_{5-j}-\cos\psi_j),\\
&\alpha_j = e^{-\mathrm{i}(\psi_{5-j}-\psi_j)}, \quad \alpha_j^* = e^{\mathrm{i}(\psi_{5-j}-\psi_j)}, \quad (j=1,2),\\
&A_{12} = \frac{B_- - \sin(\psi_4-\psi_1)\sin(\psi_3-\psi_2)}{B_+ - \sin(\psi_4-\psi_1)\sin(\psi_3-\psi_2)} = \frac{\sin\frac{\psi_2-\psi_1}{2}\sin\frac{\psi_4-\psi_3}{2}}{\sin\frac{\psi_3-\psi_1}{2}\sin\frac{\psi_4-\psi_2}{2}}\\
&B_+ = 4\sin\frac{\psi_4-\psi_1}{2}\sin\frac{\psi_3-\psi_2}{2}\left(\cos\frac{\psi_4+\psi_1-\psi_3-\psi_2}{2} + \sin\frac{\psi_4-\psi_1}{2}\sin\frac{\psi_3-\psi_2}{2}\right)\\
&B_- = 4\sin\frac{\psi_4-\psi_1}{2}\sin\frac{\psi_3-\psi_2}{2}\left(\cos\frac{\psi_4+\psi_1-\psi_3-\psi_2}{2} - \sin\frac{\psi_4-\psi_1}{2}\sin\frac{\psi_3-\psi_2}{2}\right)
\end{aligned} \tag{35}$$

The soliton interaction of this solution is called P-type and corresponds to the permutation $\pi = (4\,3\,2\,1)$. The chord diagram of the P-type 2-dark line soliton solution is given in Fig.6 (right figure).

General 2-dark line soliton solutions

By using parameters $\psi_1, \psi_2, \psi_3, \psi_4$, the general 2-dark line soliton solution is expressed in the following determinant form:

$$
\begin{aligned}
&f = \tau^{\left(-\frac{1}{2}\right)},\ g = \tau^{\left(\frac{1}{2}\right)},\ g^* = \tau^{\left(-\frac{3}{2}\right)} \\
&\tau^{(s)} = \begin{vmatrix} \varphi_1^{(s)} & \varphi_1^{(s+1)} \\ \varphi_2^{(s)} & \varphi_2^{(s+1)} \end{vmatrix}, \\
&\varphi_j^{(s)}(x,y,t) = \sum_{i=1}^{4} a_{ij} E_i^{(s)}(x,y,t), \quad E_i^{(s)}(x,y,t) = e^{\theta_i - \mathrm{i}s\psi_i}, \\
&\theta_i = (\rho_0 \sin\psi_i)x - (\rho_0\cos\psi_i)y - (2k\rho_0\sin\psi_i + 2l\rho_0\cos\psi_i + \rho_0^2\sin 2\psi_i)t, \\
&\psi_1 < \psi_2 < \psi_3 < \psi_4.
\end{aligned}
$$

The above τ-function $\tau^{(s)}$ is written as

$$
\begin{aligned}
&\tau^{(s)} = |A\ E^{(s)}|, \qquad (36) \\
&A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}, \quad E^{(s)} = \begin{pmatrix} e^{\theta_1+\mathrm{i}s\psi_1} & e^{\theta_1+\mathrm{i}(s+1)\psi_1} \\ e^{\theta_2+\mathrm{i}s\psi_2} & e^{\theta_2+\mathrm{i}(s+1)\psi_2} \\ e^{\theta_3+\mathrm{i}s\psi_3} & e^{\theta_3+\mathrm{i}(s+1)\psi_3} \\ e^{\theta_4+\mathrm{i}s\psi_4} & e^{\theta_4+\mathrm{i}(s+1)\psi_4} \end{pmatrix}.
\end{aligned}
$$

As in the case of KP II line solitons, the A-matrix and parameters $\psi_1, \psi_2, \psi_3, \psi_4$ determine 2-dark line soliton interactions of the DS II system. By the Cauchy-Binet formula, $\tau^{(s)}$ is rewritten as

$$
\begin{aligned}
\tau^{(s)} &= \sum_{1\le m_1<m_2\le 4} A(m_1,m_2)E^{(s)}(m_1,m_2) \\
&= \sum_{1\le m_1<m_2\le 4} A(m_1,m_2)e^{\mathrm{i}s\psi_{m_1}+\mathrm{i}s\psi_{m_2}}(e^{\mathrm{i}\psi_{m_2}} - e^{\mathrm{i}\psi_{m_1}})e^{\theta_{m_1}+\theta_{m_2}} \\
&= 2\mathrm{i} \sum_{1\le m_1<m_2\le 4} A(m_1,m_2)e^{\mathrm{i}\left(s+\frac{1}{2}\right)(\psi_{m_1}+\psi_{m_2})}\sin\frac{\psi_{m_2}-\psi_{m_1}}{2}e^{\theta_{m_1}+\theta_{m_2}} \\
&\sim \sum_{1\le m_1<m_2\le 4} A(m_1,m_2)e^{\mathrm{i}\left(s+\frac{1}{2}\right)(\psi_{m_1}+\psi_{m_2})}\sin\frac{\psi_{m_2}-\psi_{m_1}}{2}e^{\theta_{m_1}+\theta_{m_2}} \qquad (37)
\end{aligned}
$$

where $A(m_1, m_2)$ is the 2×2 minors of the matrix A obtained from m_1-th and m_2-th columns $(1 \le m_1 < m_2 \le 4)$ and $E(m_1, m_2)$ is the 2×2 minors of the matrix $E^{(s)}$ obtained from m_1-th and m_2-th rows $(1 \le m_1 < m_2 \le 4)$.

The A-matrix is canonically chosen in the reduced row echelon form (RREF). As in the case of the KP II equation, there are total seven different types of (non-singular) 2-soliton interactions which can be enumerated according to the seven derangements (permutations that have no fixed points) of the index set

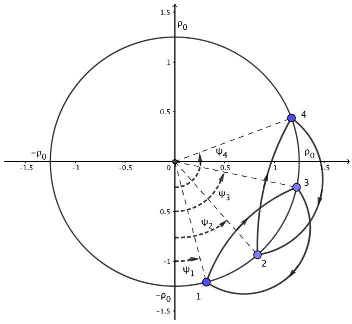


Figure 7: The chord diagram of T-type 2-dark line soliton solution.

$[1, 2, 3, 4]$ with two excedances:

$$[1]\quad \pi = (3\,4\,1\,2): \quad A = \begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix}, \; a, b, c, d > 0, ad - bc > 0, \text{ T-type}$$

$$[2]\quad \pi = (2\,4\,1\,3): \quad A = \begin{pmatrix} 1 & 0 & -c & -d \\ 0 & 1 & a & b \end{pmatrix}, \; a, b, c, d > 0, \; ad - bc = 0,$$

$$[3]\quad \pi = (4\,3\,1\,2): \quad A = \begin{pmatrix} 1 & 0 & -b & -c \\ 0 & 1 & a & 0 \end{pmatrix}, \; a, b, c > 0,$$

$$[4]\quad \pi = (3\,4\,2\,1): \quad A = \begin{pmatrix} 1 & 0 & 0 & -c \\ 0 & 1 & a & b \end{pmatrix}, \; a, b, c > 0,$$

$$[5]\quad \pi = (4\,3\,2\,1): \quad A = \begin{pmatrix} 1 & 0 & 0 & -b \\ 0 & 1 & a & 0 \end{pmatrix}, \; a, b > 0, \text{ P-type}$$

$$[6]\quad \pi = (3\,1\,4\,2): \quad A = \begin{pmatrix} 1 & a & 0 & -c \\ 0 & 0 & 1 & b \end{pmatrix}, \; a, b, c > 0,$$

$$[7]\quad \pi = (2\,1\,4\,3): \quad A = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 0 & 1 & b \end{pmatrix}, \; a, b > 0, \text{ O-type}$$

These matrices are irreducible totally non-negative (TNN) matrices, i.e., all its maximal minors are non-negative, and correspond to points on TNN Grassmannian $Gr^+(2, 4)$. The chord diagram of the T-type 2-dark line soliton solution is given in Fig.7.

General multi-dark line soliton solutions

Multi-dark line soliton solutions of the defocusing DS II system is given as follows:

$$\begin{aligned}
&f = \tau^{(\frac{1-N}{2})}, \; g = \tau^{\left(\frac{1-N}{2}+1\right)}, \; g^* = \tau^{\left(\frac{1-N}{2}-1\right)}, \\
&\tau^{(s)} = \begin{vmatrix} \varphi_1^{(s)} & \cdots & \varphi_1^{(s+N-1)} \\ \vdots & \ddots & \vdots \\ \varphi_N^{(s)} & \cdots & \varphi_N^{(s+N-1)} \end{vmatrix}, \\
&\varphi_j^{(s)}(x, y, t) = \sum_{i=1}^{M} a_{ij} E_i^{(s)}(x, y, t), \quad E_i^{(s)}(x, y, t) = e^{\theta_i - \mathrm{i} s \psi_i}, \\
&\theta_i = (\rho_0 \sin\psi_i)x - (\rho_0 \cos\psi_i)y - (2k\rho_0 \sin\psi_i + 2l\rho_0 \cos\psi_i + \rho_0^2 \sin 2\psi_i)t, \\
&\psi_1 < \psi_2 < \cdots \psi_{M-1} < \psi_M \quad (i = 1, 2, \cdots, M, \; j = 1, 2, \cdots, N).
\end{aligned} \tag{38}$$

The above τ-function $\tau^{(s)}$ is written as

$$\tau^{(s)} = |A\ E^{(s)}|, \tag{39}$$
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NM} \end{pmatrix}, \quad E^{(s)} = \begin{pmatrix} e^{\theta_1 + \mathrm{i}s\psi_1} & \cdots & e^{\theta_1 + \mathrm{i}(s+N-1)\psi_1} \\ \vdots & \ddots & \vdots \\ e^{\theta_M + \mathrm{i}s\psi_M} & \cdots & e^{\theta_M + \mathrm{i}(s+N-1)\psi_M} \end{pmatrix}.$$

As in the case of KP II line solitons, the A-matrix and the parameters $\psi_1, \psi_2, \cdots, \psi_M$ determine dark line soliton interactions of the DS II system. By the Cauchy-Binet formula, $\tau^{(s)}$ is rewritten as

$$\begin{aligned}
\tau^{(s)} &= \sum_{1 \le m_1 < \cdots < m_N \le M} A(m_1, \cdots, m_N) E^{(s)}(m_1, \cdots, m_N) \\
&= \sum_{1 \le m_1 < \cdots < m_N \le M} A(m_1, \cdots, m_N) e^{\sum_{j=1}^N \mathrm{i}s\psi_{m_j}} \prod_{1 \le i_1 < i_2 \le N} (e^{\mathrm{i}\psi_{m_{i_2}}} - e^{\mathrm{i}\psi_{m_{i_1}}}) e^{\sum_{j=1}^N \theta_{m_j}} \\
&= (2\mathrm{i})^{\frac{N(N-1)}{2}} \sum_{1 \le m_1 < \cdots < m_N \le M} A(m_1, \cdots, m_N) e^{\sum_{j=1}^N \mathrm{i}\left(s + \frac{N-1}{2}\right)\psi_{m_j}} \prod_{1 \le i_1 < i_2 \le N} \sin \frac{\psi_{m_{i_2}} - \psi_{m_{i_1}}}{2} e^{\sum_{j=1}^N \theta_{m_j}} \\
&\sim \sum_{1 \le m_1 < \cdots < m_N \le M} A(m_1, \cdots, m_N) e^{\sum_{j=1}^N \mathrm{i}\left(s + \frac{N-1}{2}\right)\psi_{m_j}} \prod_{1 \le i_1 < i_2 \le N} \sin \frac{\psi_{m_{i_2}} - \psi_{m_{i_1}}}{2} e^{\sum_{j=1}^N \theta_{m_j}}
\end{aligned} \tag{40}$$

where $A(m_1, \cdots, m_N)$ is the $N \times N$ minors of the matrix A obtained from m_1-th,...,m_N-th columns ($1 \le m_1 < \cdots < m_N \le M$) and $E(m_1, \cdots, m_N)$ is the $N \times N$ minors of the matrix $E^{(s)}$ obtained from m_1-th, ..., m_N-th rows ($1 \le m_1 < \cdots < m_N \le M$).

As mentioned above, DS II 1-dark line soliton of the same shape can propagate in 2 different speeds. This means that a given soliton pattern can evolve into different patterns. So the multi-dark line soliton interactions for the defocusing DS II system can be more complicated compared to the KP II equation. However we can analyze dark line soliton interactions by using the tools developed in the case of the KP II equation because 2 dark line solitons with the same shape and different speeds can be expressed in 2 different chord diagrams (i.e., points corresponding to ψs on the circle are different for these 2 dark line solitons).

By using the chord diagrams presented in this article and triangulations, we can analyze the interactions of multi-dark line solitons for the DS II system. The angle dependency of soliton interactions discussed in the previous section can be explained by using chord diagrams.

As in the KP II line solitons [9], we can solve an inverse problem for the DS II dark line solitons in which the τ-function is determined and the time evolution is predicted from a given initial pattern. The algorithm to solve the inverse problem is as follows: i) From an initial soliton pattern, get data of angles, depth and speed of line solitons, ii) from these data, construct the data of the parameters ψ_i, iii) draw a chord diagram corresponding to an initial pattern, iv) using the chord diagram, we construct the A-matrix and τ-function.

The details of these issues will be addressed in our forthcoming paper.

5 Conclusions

In this article, we have studied the interactions of the dark line solitons for the defocusing DS II system. We have investigated the angle dependency of 2-dark line soliton interactions and confirmed that the numerical experiments match with the analytical results by using the exact 2-dark line soliton solution. We have also given the general multi-dark line soliton solution with full parameters in the Wronskian form and the chord diagrams for the DS II system. The detail of the analysis of DS II dark line soliton interactions will be given in our forthcoming paper. We would like to thank Prof. Yuji Kodama and Prof. Sarbarish Chakravarty for useful discussions. This work was supported by JSPS KAKENHI Grant Number JP18K03435, JP17H02856 and the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

References

[1] G. Biondini and Y. Kodama, "On a family of solutions of the Kadomtsev-Petviashvili equation which also satisfy the Toda lattice hierarchy", J. Phys. A: Math. Gen., **36** (2003) 10519.

[2] Y. Kodama, "Young diagrams and N-soliton solutions of the KP equation", J. Phys. A: Math. Gen., **37** (2004) 11169–11190.

[3] G. Biondini and S. Chakravarty, "Soliton solutions of the Kadomtsev-Petviashvili II equation", J. Math. Phys., **47** (2006) 033514.

[4] S. Chakravarty and Y. Kodama, "Classification of the line-soliton solutions of KP II", J. Phys. A: Math. Theor., **41** (2008) 275209.

[5] S. Chakravarty and Y. Kodama, "Soliton solutions of the KP equation and application to shallow water waves", Stud. Appl. Math., **123** (2009) 83–151.

[6] Y. Kodama, M. Oikawa, and H. Tsuji, "Soliton solutions of the KP equation with V-shape initial waves", J. Phys. A: Math. Theor., **42** (2009) 312001.

[7] Y. Kodama, "KP solitons in shallow water", J. Phys. A: Math. Theor., **43** (2010) 434004.

[8] C-Y. Kao and Y. Kodama, "Numerical study of the KP equation for non-periodic waves", Math. Comp. Sim., **82** (2012) 1185–1218.

[9] S. Chakravarty and Y. Kodama, "Construction of KP solitons from wave patterns", J. Phys. A:Math. Theor., **47** (2013) 025201.

[10] Y. Kodama, "KP Solitons and the Grassmannians: combinatorics and geometry of two-dimensinal wave patterns", Springer Briefs in Mathematical Physics, **22** (2017).

[11] A. Davey and K. Stewartson, "On three-dimensional packets of surface waves", Proc. Roy. Soc. Lond. A, **338** (1974) 101–110.

[12] D. J. Benney and G. J. Roskes, "Wave instabilities", Stud. Appl. Math., **48** (1969) 377–385.

[13] V. D. Djordjevic and L. G. Redekopp, "On two-dimensional packets of capillary-gravity waves", J. Fluid Mech., **79** (1977) 703-714.

[14] M. J. Ablowitz and H. Segur, "On the evolution of packets of water waves", J. Fluid Mech. **92** (1979) 92.

[15] M. J. Ablowitz and P. A. Clarkson, "Solitons, Nonlinear Evolution Equations and Inverse Scattering", (Camb. Univ. Press, 1991).

[16] M. J. Ablowitz, "Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons", (Camb. Univ. Press, 2011).

[17] F. Kako and N. Yajima, "Interaction of ion-acoustic solitons in two-dimensional space", J. Phys. Soc. Jpn., **49** (1980) 2063–2071.

[18] P. W. White and J. A. C. Weideman, "Numerical simulation of solitons and dromions in the Davey-Stewartson system", Math. Comp. Sim., **37** (1994) 469–479.

[19] P. Schlatter, N.A. Adams, and L. Kleiser, "A windowing method for periodic inflow/outflow boundary treatment of non-periodic flows", J. Comp. Phys., **206** (2005) 505–535.

[20] K. Nishinari, K. Abe, and J. Satsuma, "A New-Type of Soliton Behavior in a Two Dimensional Plasma System", J. Phys. Soc. Jpn., **62** (1993) 2021–2029.